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NUMERICAL BOUNDS ON DIVERGENCES IN TERMS OF SYMMETRIC CHI-SQUARE DIVERGENCE MEASURE

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Abstract: Information divergence measures are well known in the literature of Information Theory and Statistics. Information divergence measures play an important role in pattern recognition and information retrieval. In this paper we will establish an upper and lower bounds of Relative Jenson-Shannon divergence, Relative arithmetic –geometric divergence, Kullback-Leibler divergence and Triangular discrimination in terms of Symmetric chi-square divergence measure using a new f-divergence measure and inequalities. Numerical bounds of well-known divergence measures are also studied.

Keywords: Kullback-Leibler divergence measure, new f- divergence, Symmetric Chi-square divergence etc.

1. INTRODUCTION

Let

$$\Gamma_n = \{p = (p_1, p_2, p_3, ...p_n) : p_i \geq 0, \sum_{i=1}^{n} p_i = 1, \ \ n \geq 2\} \qquad 1.1$$

be the set of complete finite discrete probability distributions. There are many information and divergence measures exist in the literature on information theory and statistics. In this section we present some properties of new f-divergence measure introduced in Jain & Saraswat [7] & [8] and its particular cases which are interesting in areas of information theory and statistics is given by

$$S_f(P,Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i + q_i}{2q_i}\right) \qquad 1.2$$

Where $f : \Re_+ \to \Re_+$ is a convex function and P, Q ϵ Γ_n.

Proposition 1.1 Let $f : [0, \infty) \to \Re$ be convex and $P, Q \in \Gamma_n$ with $P_n = Q_n = 1$ then we have the following inequality

$$S_f(P,Q) \geq f(1) \qquad 1.3$$

Equality holds in (1.3) iff

$$p_i = q_i, \forall i = 1, 2, ..., n \qquad 1.4$$

Corollary 1.1.1 (Non-Negativity of New f-divergence measure) Let $f : [0, \infty) \to \Re$ be convex and normalized i.e.

$$f(1)=0 \qquad 1.5$$

Then for any $P, Q \in \Gamma_n$ from (1.3) of proposition 1.1 and (1.5), we have the inequality

$$S_f(P,Q) \geq 0 \qquad 1.6$$

If *f* is strictly convex, equality holds in (1.6) iff

$$p_i = q_i, \forall i = 1, 2, ..., n \qquad 1.7$$

and

$$S_f(P,Q) = 0 \text{ iff } P = Q \qquad 1.8$$

Proposition 1.2 Let f_1 and f_2 are two convex functions and $g = af_1 + bf_2$ then $S_g(P,Q) = aS_{f_1}(P,Q) + bS_{f_2}(P,Q)$, Where $P, Q \in \Gamma_n$.

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It is shown that we shall derive some well-known divergence measures using new f-divergence measure such as Relative Jenson-Shannon divergence [11], Relative Arithmetic-Geometric divergence [12], Kullback-Leibler divergence [4] and Triangular discrimination [1], Chi- square divergence measure [9], Symmetric Chi- square divergence measure [2]. In this section an inequality of new f-divergence in terms of Symmetric Chi- square divergence measure is established in section 3. Using the inequality of section 3, bounds of various particular measures are found in terms of Symmetric chi-square divergence measure in section 4. Numerical bounds of some well-known divergence measures are discussed in section 5.

We now give some examples of well-known information divergence measures which are obtained from new f-divergence measure.

- If $f(t)=(2t-1)\log(2t-1), \forall t>\frac{1}{2}$ then Kullback-Leibler divergence measure is given by

$$S_f(P,Q)=\sum_{i=1}^{n} p_i \log\frac{p_i}{q_i}=KL(P,Q) \quad 1.9$$

- If $f(t)=(t-1)^2$ then Chi-square divergence measure is given by

$$S_f(P,Q)=\frac{1}{4}\left[\sum_{i=1}^{n}\frac{p_i^2}{q_i}-1\right]=\frac{1}{4}\chi^2(P,Q) \quad 1.10$$

- If $f(t)=\frac{t(t-1)^2}{(2t-1)}, \forall t>\frac{1}{2}$ then symmetric chi-square divergence is given by

$$S_f(P,Q)=\frac{1}{8}\left[\sum_{i=1}^{n}\frac{(p_i+q_i)(p_i-q_i)^2}{p_i q_i}\right]=\frac{1}{8}\Psi(P,Q) \quad 1.11$$

- If $f(t)=-\log t$ then relative Jensen-Shannon divergence measure is given by

$$S_f(P,Q)=\sum_{i=1}^{n} q_i \log\left(\frac{2q_i}{p_i+q_i}\right)=F(Q,P) \quad 1.12$$

- If $f(t)=t\log t$ then relative arithmetic-geometric divergence measure is given by

$$S_f(P,Q)=\sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2q_i}\right)=G(Q,P) \quad 1.13$$

- If $f(t)=\frac{(t-1)^2}{t}, \forall t>0$ then Triangular discrimination is given by

$$S_f(P,Q)=\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{2(p_i+q_i)}=\frac{1}{2}\Delta(P,Q) \quad 1.14$$

2. SYMMETRIC CHI-SQUARE DIVERGENCE MEASURE

Let $f(t)$ be a convex function and normalized i.e. $f(1)=0$; which is given by

$$f(t)=\frac{t(1-t)^2}{2t-1}, \forall t>\frac{1}{2} \quad 2.1$$

$$f'(t)=\left[\frac{4t^3-7t^2+4t-1}{(2t-1)^2}\right] \quad 2.2$$

$$f''(t)=\left[1+\frac{1}{(2t-1)^3}\right]>0, \forall t>\frac{1}{2} \quad 2.3$$

$f''(t) > 0$. Hence function is convex.

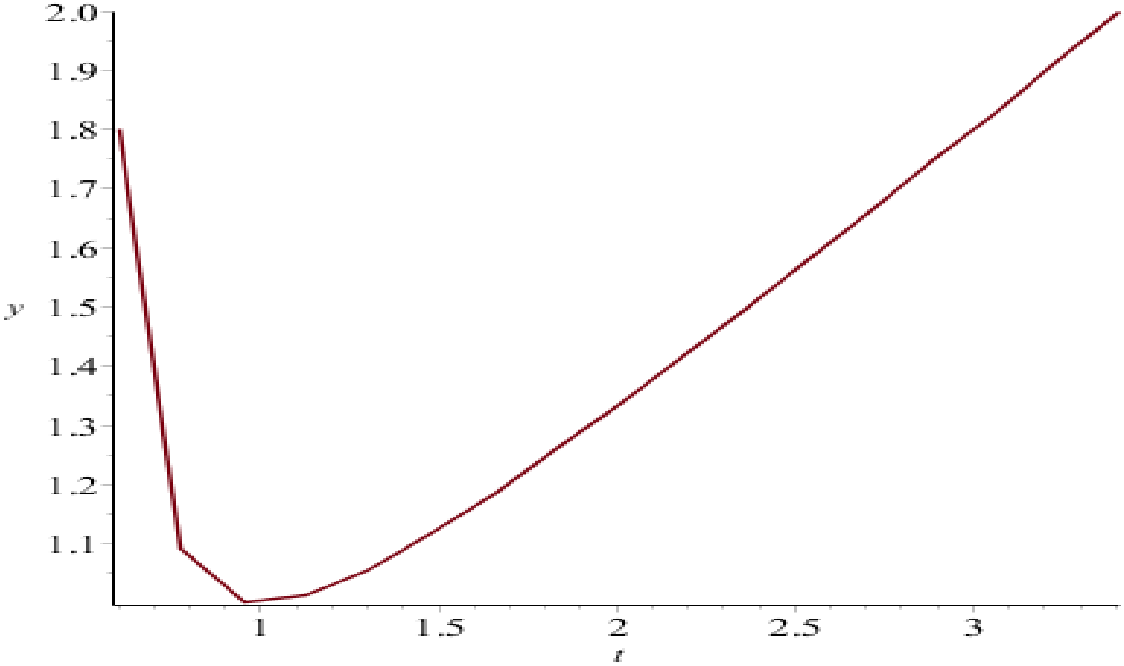

Fig. 2.1Graphical representation of $f(t)$

Applying properties of new f-divergence measure on $f(t)$, we get

$$\sum_{i=1}^{n} q_i \frac{\left(\frac{p_i+q_i}{2q_i}\right)\left(\frac{p_i+q_i}{2q_i}-1\right)^2}{2\left(\frac{p_i+q_i}{2q_i}\right)-1} \quad 2.4$$

$$\sum_{i=1}^{n} \frac{(p_i+q_i)(p_i-q_i)^2}{8p_iq_i} = \frac{1}{8}\Psi(P,Q) \quad 2.5$$

$$S_f(P,Q) = \frac{1}{8}\Psi(P,Q) \quad 2.6$$

Where $\Psi(P,Q)$ is Symmetric chi-square divergence measure

3. NEW INFORMATION INEQUALITY

The following theorem concerning an upper and lower bound for a new f-divergence measure in terms of the Symmetric Chi-Square divergence measure holds.

The result are on similar lines to the result presented by Dragomir[1] and Jain and Saraswat[4-5].

Theorem 3.1: Let us consider the generating mapping $F_m : \left(\frac{1}{2}, \infty\right) \to \Re$ is normalized i.e. *f*(1)=0 and satisfies the assumptions.

(i) *f* is twice differentiable on (r, R), where $0 \le r \le 1 \le R \le \infty$

(ii) There exist constants m, M such that

$$m \le \left[\frac{\{(2t-1)^3 f''(t)\}}{\{(2t-1)^3+1\}}\right] \le M, \forall t \in (r,R). \quad 3.1$$

If P, Q are discrete probability distributions satisfying the assumptions

$$r \le r_i = \frac{(p_i+q_i)}{2q_i} \le R < \infty, \forall i \in \{1,2,3........n\} \quad 3.2$$

Then we have the inequality

$$\frac{m}{8}\Psi(P,Q)\le S_f(P,Q)\le\frac{M}{8}\Psi(P,Q) \qquad 3.3$$

Proof: Define a mapping $F_m:\left(\frac{1}{2},\infty\right)\to\Re, F_m(t)-m\left(\frac{t(t-1)^2}{2t-1}\right),\forall t>\frac{1}{2}.$

Then $F_m(.)$ is normalized and twice differentiable, since

$$F_m''(t)=f''(t)-m\left[\frac{\{(2t-1)^3+1\}}{(2t-1)^3}\right]$$

$$F_m''(t)=\frac{\{(2t-1)^3+1\}}{(2t-1)^3}\left[\frac{(2t-1)^3}{\{(2t-1)^3+1\}}f''(t)-m\right]\ge 0 \qquad 3.4$$

For all t $\in$ (r, R), it follows that $F_m(.)$ is convex on (r, R). Applying non-negativity property of *f*-divergence functional for $F_m(.)$ and by proposition 1.2, we may state that

$$0\le S_{F_m}(P,Q)=S_f(P,Q)-mS_{\frac{t(t-1)^2}{2t-1}}(P,Q)=S_f(P,Q)-\frac{m}{8}\Psi(P,Q)$$

$$\Rightarrow 0\le S_f(P,Q)-\frac{m}{8}\Psi(P,Q) \qquad 3.5$$

From where the first inequality of (3.3) results.

Now we again Define a mapping $F_M:\left(\frac{1}{2},\infty\right)\to\Re, F_M(t)=M\left(\frac{t(t-1)^2}{2t-1}\right)-f(t)$, which is obviously normalized, twice differentiable and by (3.1), convex on (r, R). Applying non-negativity property of f-divergence functional for $F_m(.)$ and Proposition 1.2, we obtain the second part of (3.3) i.e.

$$0\le\frac{M}{8}\Psi(P,Q)-S_f(P,Q) \qquad 3.6$$

From (3.5) and (3.6) give the result (3.3).

4. RESULTS

In this section we established bounds of particular well known divergence measures in terms of Symmetric Chi-Square divergence using inequality of (3.3) of Theorem 3.1 which may be interested in Information Theory and statistics.

The result is on similar lines to the result presented by Dragomir [1] and Jain & Saraswat [4-5].

Proposition 4.1: Let $P,Q\in\Gamma_n$ be two probability distributions with the property that

$$r\le r_i=\frac{(p_i+q_i)}{2q_i}\le R<\infty,\forall i\in\{1,2,3.........n\}$$

Then we have the following inequality

$$\frac{1}{8}\min\left[\frac{(2r-1)^3}{\left\{(2r-1)^3+1\right\}r^2},\frac{(2R-1)^3}{\left\{(2R-1)^3+1\right\}R^2}\right]\Psi(P,Q)\le F(Q,P)\le(0.0654)\Psi(P,Q) \qquad 4.1$$

Proof: Consider the mapping $f:(0,\infty)\to\Re$

$$f(t)=-\log t, f'(t)=\left(\frac{-1}{t}\right), f''(t)=\left(\frac{1}{t^2}\right)>0,\forall t>0$$

So function is convex and normalized i.e. $f(1)=0$

Define $$g:[r,R]\to\Re,\ g(t)=\left[\frac{(2t-1)^3 f''(t)}{\{(2t-1)^3+1\}}\right]$$

$$g(t)=\left[\frac{(2t-1)^3}{\{(2t-1)^3+1\}t^2}\right]$$

$$g(t)=\left[\frac{1}{t^2}-\frac{1}{(8t^5-12t^4+6t^3)}\right]$$

$$g'(t)=\left[\frac{-2}{t^3}+\frac{(40t^4-48t^3+18t^2)}{(8t^5-12t^4+6t^3)^2}\right],$$

$$g'(t)=0,\left(t_0=\frac{1}{2},\frac{1}{2},1.1020\approx 1.10\right)$$

$$g''(t)=\left[\frac{6}{t^4}-\frac{2(40t^4-48t^3+18t^2)^2}{(8t^5-12t^4+6t^3)^3}+\frac{(160t^3-144t^2+36t)}{(8t^5-12t^4+6t^3)^2}\right]$$

$g''(t)<0$ at $t=1.10$, It is maximum

It is clear that g(t) is monotonic increasing on $\left(\frac{1}{2},1.1\right]$ and monotonic decreasing on $\left(1.10,\infty\right)$ which shows that function g(t) has minimum realized at $t_0=1.1020\approx 1.10$ .Then obviously

$M=\sup_{t\in[r,R]} g(t)=g(R)=g(1.10)=0.5234$,

$$m=\inf_{t\in[r,R]} g(t)=\min\left[g(r),g(R)\right]=\min\left[\frac{(2r-1)^3}{\{(2r-1)^3+1\}r^2},\frac{(2R-1)^3}{\left\{(2R-1)^3+1\right\}R^2}\right] \quad 4.2$$

Also $S_f(P,Q)=\sum q_i\log\left(\frac{2q_i}{p_i+q_i}\right)=F(Q,P)$ from equation (1.9),(3.3)&(4.2)

Then We get inequality (4.1)

Proposition 4.2: Let $P,Q\in\Gamma_n$ be two probability distributions satisfying (3.2)

Then we have the following inequality

$$\frac{1}{8}\min\left(\frac{(2r-1)^3}{\{(2r-1)^3+1\}r},\frac{(2R-1)^3}{\{(2R-1)^3+1\}R}\right)\Psi(P,Q)\le G(Q,P)\le\left(0.0772\right)\Psi(P,Q) \quad 4.3$$

Proof: consider the mapping $f:\left(0,\infty\right)\to\Re$

$$f(t)=t\log t,\ f'(t)=1+\log t,\ f''(t)=(1/t)>0,\forall t>0$$

So function is convex and normalized i.e. $f(1)=0$

Define $g:[r,R]\to\Re,$

$$g(t)=\left[\frac{(2t-1)^3 f''(t)}{\{(2t-1)^3+1\}}\right]$$

$$g(t)=\left[\frac{(2t-1)^3}{\{(2t-1)^3+1\}t}\right]$$

$$g(t)=\left[\frac{1}{t}-\frac{1}{(8t^4-12t^3+6t^2)}\right]$$

$$g'(t)=\left[\frac{-1}{t^2}+\frac{(32t^3-36t^2+12t)}{(8t^4-12t^3+6t^2)^2}\right],\quad g'(t)=0,\left(t=\frac{1}{2},\frac{1}{2},1.2873\approx 1.28\right)$$

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$$g''(t) = \left[\frac{2}{t^3} - \frac{2(32t^3 - 36t^2 + 12t)^2}{(8t^4 - 12t^3 + 6t^2)^3} + \frac{(96t^2 - 72t + 12)}{(8t^4 - 12t^3 + 6t^2)^2} \right]$$

$g''(t) < 0$, at $t = 1.28$, It is maximum.

It is clear that g(t) is monotonic increasing on $\left(\frac{1}{2}, 1.28\right]$ and monotonic decreasing on $(1.28, \infty)$.

Which shows that function g(t) has the maximum realized at $t_0 = 1.2873 \approx 1.28$

Then obviously

$M = \sup_{t\in[r,R]} g(t) = g(R) = g(1.28) = 0.6183$

$$m = \inf_{t\in[r,R]} g(t) = \min\left[g(r), g(R)\right] = \min\left[\frac{(2r-1)^3}{\{(2r-1)^3+1\}r}, \frac{(2R-1)^3}{\left\{(2R-1)^3+1\right\}R} \right] \quad 4.4$$

Also $S_f(P,Q) = G(Q,P)$

From equation (1.10), (3.3) & (4.4) then We get result (4.3)

Proposition 4.3: Let $P, Q \in \Gamma_n$ be two probability distributions satisfying (3.2),

Then we have the following inequality

$$\frac{1}{8}\left(\frac{4(2r-1)^2}{\left\{(2r-1)^3+1\right\}}, \frac{4(2R-1)^2}{\{(2R-1)^3+1\}} \right)\Psi(P,Q) \le KL(P,Q) \le (0.2645)\Psi(P,Q) \quad 4.5$$

Proof: Consider the mapping $f : \left(\frac{1}{2}, \infty\right) \to \Re$

$$f(t) = (2t-1)\log(2t-1),\ f'(t) = 2\{1+\log(2t-1)\},\ f''(t) = \frac{4}{(2t-1)} > 0, \forall t > 0$$

So function is convex and normalized i.e. $f(1) = 0$

Define $g : [r, R] \to \Re$,

$$g(t) = \left[\frac{(2t-1)^3 f''(t)}{\{(2t-1)^3+1\}} \right]$$

$$g(t) = \frac{4(2t-1)^2}{(2t-1)^3+1}$$

$$g'(t) = \frac{16(2t-1)}{(2t-1)^3+1} - \frac{24(2t-1)^4}{((2t-1)^3+1)^2}$$

$$g'(t) = 0, \left(t = \frac{1}{2}, \frac{1}{2}(2)^{\frac{1}{3}} + \frac{1}{2} = 1.1299 \approx 1.13 \right)$$

$$g''(t) = \frac{32}{(2t-1)^3+1} - \frac{288(2t-1)^3}{((2t-1)^3+1)^2} + \frac{288(2t-1)^6}{((2t-1)^3+1)^3}$$

$g''(t) < 0$, at $t = 1.13$, So It is maximum.

It is clear that g(t) is monotonic increasing on $\left(\frac{1}{2}, 1.13\right]$ and monotonic decreasing on $(1.13, \infty)$. Which shows that function g(t) has the maximum realized at $t_0 = 1.1299 \approx 1.13$ we have two cases :

Then obviously

$M = \sup_{t\in[r,R]} g(t) = g(R) = g(1.12) = 2.1165$

$$m = \inf_{t\in[r,R]} g(t) = \min\left[g(r), g(R)\right] = \min\left[\frac{4(2r-1)^2}{\{(2r-1)^3+1\}}, \frac{4(2R-1)^2}{\left\{(2R-1)^3+1\right\}} \right] \quad 4.6$$

Also $S_f(P,Q) = KL(P,Q)$

From equation (1.11), (3.3) & (4.6) give the result (4.5)

Proposition 4.4: Let $P,Q \in \Gamma_n$ be two probability distributions satisfying (3.2),

Then we have the following inequality

$$\frac{1}{8}\min\left[\frac{(2r-1)^3}{\left\{(2r-1)^3+1\right\}}\left(\frac{2}{r}-\frac{4(r-1)}{r^2}+\frac{2(r-1)^2}{r^3}\right),\frac{(2R-1)^3}{\left\{(2R-1)^3+1\right\}}\left(\frac{2}{R}-\frac{4(R-1)}{R^2}+\frac{2(R-1)^2}{R^3}\right)\right]\Psi(P,Q)$$

$$\le \frac{1}{2}\Delta(P,Q) \le (0.1250)\Psi(P,Q) \qquad 4.7$$

Proof: Consider the mapping $f:(0,\infty)\to\Re$

$$f(t)=\frac{(t-1)^2}{t},\ f'(t)=\frac{2(t-1)}{t}-\frac{(t-1)^2}{t^2},\ f''(t)=\left(\frac{2}{t}-\frac{4(t-1)}{t^2}+\frac{2(t-1)^2}{t^3}\right)>0, \forall t>0$$

So function is convex and normalized i.e. $f(1)=0$

Define $g:[r,R]\to\Re$,

$$g(t)=\left[\frac{(2t-1)^3 f''(t)}{\{(2t-1)^3+1\}}\right]$$

$$g(t)=\frac{(2t-1)^3}{\{(2t-1)^3+1\}}\left(\frac{2}{t}-\frac{4(t-1)}{t^2}+\frac{2(t-1)^2}{t^3}\right)$$

$$g'(t)=\frac{\left\{6(2t-1)^2\left(\frac{2}{t}-\frac{4(t-1)}{t^2}+\frac{2(t-1)^2}{t^3}\right)\right\}}{\left((2t-1)^3+1\right)}$$

$$+\frac{\left\{(2t-1)^3\left(\frac{-6}{t^2}+\frac{12(t-1)}{t^3}-\frac{6(t-1)^2}{t^4}\right)\right\}}{\left((2t-1)^3+1\right)}$$

$$-\frac{\left\{6(2t-1)^5\left(\frac{2}{t}-\frac{4(t-1)}{t^2}+\frac{2(t-1)^2}{t^3}\right)\right\}}{\left((2t-1)^3+1\right)}$$

$$g'(t)=0,(t=\tfrac{1}{2},1),$$

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$$g''(t)=\frac{\left\{24(2t-1)\left(\frac{2}{t}-\frac{4(t-1)}{t^2}+\frac{2(t-1)^2}{t^3}\right)\right\}}{\left((2t-1)^3+1\right)}$$
$$+\frac{\left\{12(2t-1)^2\left(\frac{-6}{t^2}+\frac{12(t-1)}{t^3}-\frac{6(t-1)^2}{t^4}\right)\right\}}{\left((2t-1)^3+1\right)}$$
$$-\frac{\left\{96(2t-1)^4\left(\frac{2}{t}-\frac{4(t-1)}{t^2}+\frac{2(t-1)^2}{t^3}\right)\right\}}{\left((2t-1)^3+1\right)^2}$$
$$+\frac{\left\{(2t-1)^3\left(\frac{24}{t^3}-\frac{48(t-1)}{t^4}+\frac{24(t-1)^2}{t^5}\right)\right\}}{\left((2t-1)^3+1\right)}$$
$$-\frac{\left\{12(2t-1)^5\left(\frac{-6}{t^2}+\frac{12(t-1)}{t^3}-\frac{6(t-1)^2}{t^4}\right)\right\}}{\left((2t-1)^3+1\right)^2}$$
$$+\frac{\left\{72(2t-1)^7\left(\frac{2}{t}-\frac{4(t-1)}{t^2}+\frac{2(t-1)^2}{t^3}\right)\right\}}{\left((2t-1)^3+1\right)^3}$$

$g''(t)<0,$ at $t=1,$ It is maximum.

It is clear that g(t) is monotonic increasing on $\left(\frac{1}{2},1\right]$ and monotonic decreasing on $\left(1,\infty\right)$. which shows that function g(t) has the maximum realized at $t_0=1$.

Then obviously

$$M=\sup_{t\in[r,R]} g(t)=g(R)=g(1)=1$$

$$m=\inf_{t\in[r,R]} g(t)=\min\left[g(r),g(R)\right]$$

$$=\left[\frac{(2r-1)^3}{\left\{(2r-1)^3+1\right\}}\left(\frac{2}{r}-\frac{4(r-1)}{r^2}+\frac{2(r-1)^2}{r^3}\right),\frac{(2R-1)^3}{\left\{(2R-1)^3+1\right\}}\left(\frac{2}{R}-\frac{4(R-1)}{R^2}+\frac{2(R-1)^2}{R^3}\right)\right] \quad 4.8$$

Also $S_f(P,Q)=\frac{1}{2}\Delta(P,Q)$

From equation (1.12), (3.3) & (4.8) give the result (4.7).

5. NUMERICAL ILLUSTRATION

In this section, we shall discuss the numerical bounds on Relative Jensen-Shannon divergence $F(Q,P)$, Relative Arithmetic-Geometric divergence $G(Q,P)$, Kullback-Leibler divergence $KL(P,Q)$ and Triangular discrimination $\Delta(P,Q)$ in terms of Symmetric chi-Square divergence measure using following table and results (4.1), (4.3), (4.5) and (4.7).

Let P be the binomial probability distribution for the random valuable X with parameter (n=8 p=0.5) and Q its approximated normal probability distribution. The following table has given in [7] & [10].

Table 5.1 Binomial Probability Distribution (n=8 p=0.5)

x	0	1	2	3	4	5	6	7	8
p (x)	0.004	0.031	0.109	0.219	0.274	0.219	0.109	0.031	0.004
q (x)	0.005	0.030	0.104	0.220	0.282	0.220	0.104	0.030	0.005
$\frac{p(x)+q(x)}{2q(x)}$	0.900	1.016	1.0240	0.997	0.985	0.997	1.024	1.016	0.900

Here we simplified the values of $r = 0.9,\ R = 1.024$

$$\frac{1}{8}\min\left[\frac{(2(0.9)-1)^3}{\left\{(2(0.9)-1)^3+1\right\}(0.9)^2}, \frac{(2(1.024)-1)^3}{\left\{(2(1.024)-1)^3+1\right\}(1.024)^2}\right]\Psi(P,Q)$$

$$\leq F(Q,P) \leq \frac{1}{8}(0.5234)\Psi(P,Q)$$

$$\frac{1}{8}\min\left[0.4180, 0.5103\right]\Psi(P,Q) \leq F(Q,P) \leq \frac{1}{8}(0.5234)\Psi(P,Q)$$

$$(0.0522)\Psi(P,Q) \leq F(Q,P) \leq (0.0654)\Psi(P,Q) \qquad 5.1$$

$$\frac{1}{8}\min\left(\frac{(2(0.9)-1)^3}{\{(2(0.9)-1)^3+1\}(0.9)}, \frac{(2(1.024)-1)^3}{\{(2(1.024)-1)^3+1\}(1.024)}\right)\Psi(P,Q)$$

$$\leq G(Q,P) \leq \frac{1}{8}(0.6183)\Psi(P,Q)$$

$$\frac{1}{8}\min\left[0.3762, 0.5225\right]\Psi(P,Q) \leq G(Q,P) \leq \frac{1}{8}\left(0.6183\right)\Psi(P,Q)$$

$$\left(0.0470\right)\Psi(P,Q) \leq G(Q,P) \leq \left(0.0772\right)\Psi(P,Q) \qquad 5.2$$

$$\frac{1}{8}\left(\frac{4(2(0.9)-1)^2}{\left\{(2(0.9)-1)^3+1\right\}}, \frac{4(2(1.024)-1)^2}{\{(2(1.024)-1)^3+1\}}\right)\Psi(P,Q)$$

$$\leq KL(P,Q) \leq \frac{1}{8}(2.1165)\Psi(P,Q)$$

$$\frac{1}{8}\min\left[1.6931, 2.0423\right]\Psi(P,Q) \leq KL(P,Q) \leq \frac{1}{8}(2.1165)\Psi(P,Q)$$

$$(0.2116)\Psi(P,Q) \leq KL(P,Q) \leq (0.2645)\Psi(P,Q) \qquad 5.3$$

$$\frac{1}{8}\min\left[\frac{(2(0.9)-1)^3}{\left\{(2(0.9)-1)^3+1\right\}}\left(\frac{2}{(0.9)}-\frac{4(0.9-1)}{(0.9)^2}+\frac{2(0.9-1)^2}{(0.9)^3}\right), \frac{(2(1.024)-1)^3}{\left\{(2(1.024)-1)^3+1\right\}}\left(\frac{2}{1.024}-\frac{4(1.024-1)}{(1.024)^2}+\frac{2(1.024-1)^2}{(1.024)^3}\right)\right]\Psi(P,Q)$$

$$\leq \frac{1}{2}\Delta(P,Q) \leq \frac{1}{8}\Psi(P,Q)$$

$$\frac{1}{8}\min\left[0.9290, 0.9967\right]\Psi(P,Q) \leq \frac{1}{2}\Delta(P,Q) \leq \frac{1}{8}\Psi(P,Q)$$

$$(0.1161)\Psi(P,Q) \le \frac{1}{2}\Delta(P,Q) \le (0.1250)\Psi(P,Q) \qquad 5.4$$

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