ISSN Number: 2454-2024(Online)

International Journal of Technical Research & Science

ON DISTRIBUTION FUNCTION OF GENERALIZED ORDER STATISTICS FROM THE LOMAX-WEIBULL DISTRIBUTION

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Abstract- The Lomax-Weibull distribution [11] was defined by combining the cumulative distribution functions of well-known Lomax and Weibull distributions. In this paper, we derive the joint distribution function of two generalized order statistics (GOS) when the independent random variables are from continuous Lomax-Weibull distribution. Then the joint distribution of two order statistics from Lomax-Weibull distribution is derived as a special case of the main result.

Keywords: Lomax-Weibull distribution, joint distribution, generalized order statistics, cumulative distribution.

MSC Subject Classification: 60E, 62N.

1. INTRODUCTION

Order statistics finds many important applications in statistics and statistical modelling of various processes such as robust location estimates, detection of outliers, censored sampling and reliability to mention a few. These models describe random variables arranged in increasing order of magnitude.

If the random variables $X_1, X_2, \cdots X_n$ are arranged in increasing order of magnitudes and then written as $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$, then $X_{(i)}$ is called the i^{th} order statistics where i = 1, · · · , n. The unordered random variables X_i are generally statistically independent and having identical distribution but the ordered random variables X(i), i = 1, 2, · · · , n are need to be dependent.

Generalized order statistics is of interest since past few years because they are more suitable in statistical modelling of various real life problems, reliability theory and statistical inference. This concept was introduced as a unified distribution system to study various models of ordered random variables with different representations or interpretations. This concept of order statistics was further studied and extended to the concept of GOS. Kamps in his papers and books [16-18] did a pioneer work in this area. He proved how GOS generalizes the concepts of order statistics, record values, and other ordered random variables. This concept was further studied by various researchers Ahsanullah [1-3], El-Baset, Ahmed and Al-Matrofi [4], Cramer and Kamps [5-6], Cramer, Kamps and Rychlik [7-9], Kamps and Cramer [14] and Garg [10] to mention a few.

The Weibull distribution is extensively used in the modelling of data in reliability, biological studies and also in analysing life time data while the Lomax distribution is very important in modelling business problems, economics and actuarial modelling due to its heavy tail, therefore in this paper we have used the Lomax-Weibull distribution [11] which is more flexible and generalized to both Lomax and Weibull distributions.

In this paper we shall derive the joint distribution function of two GOS from the Lomax-Weibull distribution. Then the joint distribution function of order statistics from Lomax-Weibull distribution is derived as the special case of our main result.

2. PRELIMINARIES

2.1 The Lomax-Weibull distribution

The pdf of the Lomax-Weibull distribution defined by Gupta and Garg [ ] is as follows

$$f_{LW}(x;\alpha,\beta,\lambda) = \begin{cases} \dfrac{k\alpha C}{\beta\lambda}\left(\dfrac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k}\left(1+\dfrac{e^{-\left(\frac{x}{\lambda}\right)^k}}{\beta}\right)^{-(1+\alpha)} & , \quad x \geq 0 \\ 0 & , \quad x < 0 \end{cases} \tag{1}$$

And the cdf by

$$F_{LW}\left(x;\alpha,\beta,\lambda\right)=\begin{cases}1-C\left\{1-\left(1+\beta^{-1}e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)^{-\alpha}\right\} & , \quad x\geq 0\\ 0 & , \quad x<0\end{cases} \tag{2}$$

Where

$$C^{-1}=1-\left(1+\beta^{-1}\right)^{-\alpha}, \tag{3}$$

$\beta,\lambda,k>0\, and\ \alpha\neq 0$.

2.2 Generalized Order Statistics

Let us denote the cumulative density function of an absolutely continuous function with F(x) and density function with f(x) and $X_{i;n,m,k}$ and $X_{j;n,m,k}$ ($k\geq 1$, m is a real number) denote the i[th] and j[th] GOS derived from a random sample of size 'n' then the joint pdf $f_{i,j;n,m,k}(x_i,x_j)$ of i[th] and j[th] generalized order statistics is given by

$$f_{i,j;n,m,k}(x_i,x_j)=\frac{C_j}{(i-1)!(j-i-1)!}\left[1-F\left(x_i\right)\right]^{m}\left[1-F\left(x_j\right)\right]^{\gamma_j-1}\left[g_m\left(F\left(x_i\right)\right)\right]^{i-1}$$
$$\left[g_m\left(F\left(x_j\right)\right)-g_m\left(F\left(x_i\right)\right)\right]^{j-i-1} f\left(x_i\right)f\left(x_j\right) \tag{4}$$

$0\leq x_i<x_j<\infty$, $1\leq i<j\leq n$

$$C_j=\prod_{r=1}^{j}\gamma_r, \qquad \gamma_r=k+(n-r)(m+1) \tag{5}$$

$$g_m\left(x\right)=\begin{cases}\frac{1}{m+1}\left[1-\left(1-x\right)^{(m+1)}\right], & m\neq -1\\ -\ln\left(1-x\right) & , m=-1, \mathrm{x}\in(0,1)\end{cases} \tag{6}$$

Since

$$\lim_{m\to -1}\frac{1}{m+1}\left[1-\left(1-x\right)^{(m+1)}\right]=-\ln\left(1-x\right)$$

$g_m\left(x\right) = \frac{1}{m+1}\left[1-\left(1-x\right)^{(m+1)}\right]$ for all $\mathrm{x}\in(0,1)$ and for all m

With

$$g_{-1}\left(x\right) = \lim_{m\to -1} g_m\left(x\right)$$

3. THE JOINT DISTRIBUTION FUNCTION OF TWO GOS FROM LOMAX-WEIBULL DISTRIBUTION

Theorem Let $X_{i;n,m,k}$ and $X_{j;n,m,k}$ denote the i[th] and j[th] GOS from a random sample of size 'n' drawn from Lomax-Weibull distribution [1-2], then the joint pdf of the i[th] and j[th] GOS is given by

ISSN Number: 2454-2024(Online)

International Journal of Technical Research & Science

$$f_{i,j;n,m,k}(x_i,x_j)=\begin{cases}\dfrac{C_j\,\alpha^2\,\lambda^2 K^{1+m+\gamma_j+(m+1)(j-i-1)}}{\beta^2(i-1)!(j-i-1)!}\left(\dfrac{1}{m+1}\right)^{j-2}\sum_{l_1=0}^{i-1}\sum_{l_2=0}^{j-i-1}(-1)^{l_1+l_2}K^{l_1(m+1)}\binom{i-1}{l_1}\binom{j-i-1}{l_2}\left(1+\dfrac{e^{-\lambda x_i}}{\beta}\right)^{-(\alpha+1)}\left(1+\dfrac{e^{-\lambda x_j}}{\beta}\right)^{-(\alpha+1)}\\ \quad .e^{-\lambda(x_i+x_j)}\left\{1-\left(1+\dfrac{e^{-\lambda x_i}}{\beta}\right)^{-\alpha}\right\}^{m+(m+1)(l_1-l_2+j-i-1)}\left\{1-\left(1+\dfrac{e^{-\lambda x_j}}{\beta}\right)^{-\alpha}\right\}^{-\gamma_j-1+(m+1)l_2}; \ for\ m\neq-1\\ \dfrac{k^j\alpha^2\,\lambda^2\,K^{k^j}e^{-\lambda(x_i+x_j)}}{\beta^2(i-1)!(j-i-1)!}\left\{1-\left(1+\dfrac{e^{-\lambda x_i}}{\beta}\right)^{-\alpha}\right\}^{-1}\left\{1-\left(1+\dfrac{e^{-\lambda x_j}}{\beta}\right)^{-\alpha}\right\}^{k^j-1}\left(1+\dfrac{e^{-\lambda x_i}}{\beta}\right)^{-(\alpha+1)}\left(1+\dfrac{e^{-\lambda x_j}}{\beta}\right)^{-(\alpha+1)}\\ \quad .\left[-\ln K\left\{1-\left(1+\dfrac{e^{-\lambda x_i}}{\beta}\right)^{-\alpha}\right\}\right]^{(i-1)}\left[\ln\left\{1-\left(1+\dfrac{e^{-\lambda x_i}}{\beta}\right)^{-\alpha}\right\}-\ln\left\{1-\left(1+\dfrac{e^{-\lambda x_j}}{\beta}\right)^{-\alpha}\right\}\right]^{(j-i-1)}; \ for\ m=-1\end{cases} \tag{7}$$

Provided that $\alpha,\beta,\lambda>0, 1\le i<j\le n, k\ge 1$, m is a real number and n>1, $x_j>x_i\ge 0$ and C_j *and* γ_j are defined by Equation (40), and $K^{-1}=1-\left(1+\beta^{-1}\right)^{-\alpha}$.

Proof. The result can easily be established on substituting the values of f(x), F(x), and g_m(x) from Equations (1), (2) and (6) respectively in the Equation (4), expressing the values of $\left[g_m\left(F\left(x_i\right)\right)\right]^{i-1}$ and $\left[g_m\left(F\left(x_j\right)\right)-g_m\left(F\left(x_i\right)\right)\right]^{j-i-1}$ in their series forms and doing some simplification.

Special Case. If we take m=0 and $k=1$ in the above Theorem, then GOS reduce into order statistics and we get the joint distribution the i[th] and j[th] order statistics $X_{i,n}$ and $X_{j,n}$ from a sample of size n from Lomax-Weibull distribution, which is given by

$$f_{i,j;n,k}(x_i,x_j)=\frac{\alpha^2\,\lambda^2 n!K^{(n+1-i)}}{\beta^2(n-j)!(i-1)!(j-i-1)!}\sum_{l_1=0}^{i-1}\sum_{l_2=0}^{j-i-1}(-1)^{l_1+l_2}K^{l_1}\binom{i-1}{l_1}\binom{j-i-1}{l_2}\left(1+\frac{e^{-\lambda x_i}}{\beta}\right)^{-(\alpha+1)}\left(1+\frac{e^{-\lambda x_j}}{\beta}\right)^{-(\alpha+1)}$$
$$.\,e^{-\lambda(x_i+x_j)}\left\{1-\left(1+\frac{e^{-\lambda x_i}}{\beta}\right)^{-\alpha}\right\}^{(l_1-l_2+j-i-1)}\left\{1-\left(1+\frac{e^{-\lambda x_j}}{\beta}\right)^{-\alpha}\right\}^{-n+j-2+l_2} \tag{8}$$

4. ESTIMATION

We estimate the parameters of Lomax-Weibull distribution by the method of maximum likelihood. The log-likelihood for a random sample $x_1,\dots,x_n$ from pdf (1) is

$$\log L(\alpha,\beta,\lambda,k)=\log(k\,\alpha)-n\log(\lambda)-n\log(\beta)+(k-1)\sum_{i=1}^{n}\log\frac{x_i}{\lambda}-k\sum_{i=1}^{n}\frac{x_i}{\lambda}$$
$$-n\log\left\{1-\left(1+\frac{1}{\beta}\right)^{-\alpha}\right\}-(\alpha+1)\sum_{i=1}^{n}\log\left[1+\frac{1}{\beta}\exp\left(-\frac{x_i}{\lambda}\right)^k\right] \tag{9}$$

The first order partial derivatives of Log L, Equation (9), with respect to the three parameters are

$$\frac{\partial\log L}{\partial\alpha}=\frac{1}{\alpha}-\frac{n\left(1+\beta^{-1}\right)^{-\alpha}\log\left(1+\beta^{-1}\right)}{1-\left(1+\beta^{-1}\right)^{-\alpha}}-\sum_{i=1}^{n}\log\left[1+\frac{1}{\beta}\exp\left(-\frac{x_i}{\lambda}\right)^k\right] \tag{10}$$

$$\frac{\partial \log L}{\partial \beta}=-\frac{n}{\beta}+\frac{\alpha n\left(1+\beta^{-1}\right)^{-1}}{\beta^{2}\left[\left(1+\beta^{-1}\right)^{\alpha}-1\right]}+\frac{(\alpha+1)}{\beta^{2}}\sum_{i=1}^{n}\frac{\exp\left(-\frac{x_i}{\lambda}\right)^{k}}{\left[1+\beta^{-1}\exp\left(-\frac{x_i}{\lambda}\right)^{k}\right]} \quad (11)$$

$$\frac{\partial \log L}{\partial \lambda}=-\frac{n}{\lambda}-\frac{n(k-1)}{\lambda}+\sum_{i=1}^{n}\frac{x_i}{\lambda^{2}}-\frac{(\alpha+1)}{\beta\lambda^{2}}\sum_{i=1}^{n}\frac{k\,x_i\left(\frac{x_i}{\lambda}\right)^{k-1}\exp\left(-\frac{x_i}{\lambda}\right)^{k}}{\left[1+\beta^{-1}\exp\left(-\frac{x_i}{\lambda}\right)^{k}\right]} \quad (12)$$

$$\frac{\partial \log L}{\partial k}=\frac{1}{k}+\sum_{i=1}^{n}\log\frac{x_i}{\lambda}-\sum_{i=1}^{n}\frac{x_i}{\lambda}-\frac{(\alpha+1)}{\beta}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\lambda}\right)^{k}\log\left(\frac{x_i}{\lambda}\right)^{k}\exp\left(-\frac{x_i}{\lambda}\right)^{k}}{\left[1+\beta^{-1}\exp\left(-\frac{x_i}{\lambda}\right)^{k}\right]} \quad (13)$$

Setting these expressions to zero and solving them simultaneously yields the maximum-likelihood estimates of the four parameters.

CONCLUSION

Looking into the wide application, the joint distribution function of two generalized order statistics when the independent random variables are from continuous Lomax-Weibull distribution is derived. Also the joint distribution of two order statistics from Lomax-Weibull distribution is derived as a special case of the main result.

REFERENCE

[1] Ahsanullah, M. 2000. Generalised order statistics from exponential distribution. Journal of Statistical Planning and Inference. 85: 85-91.

[2] Ahsanullah, M. 2004. Record Values Theory and Application. University Press of America. London and Maryland. Ahsanullah, M. 2006. The generalised order statistics from exponential distribution. Pak. J. Statist. 22:2: 121-128.

[3] Baset, A. E., Ahmad, A. and Bakheet N. Al. Matrofi. 2006. Reccurence relation for moments and conditional moments of generalized order statistics, Univ. of Sharjah J. of Pure and Appl. Sci.. 3:3: 61-81.
Cramer, E. and Kamps, U. 2000. Relations for expectations of functions of generalized order statistics. J. Statist. Plann. and Inference. 89:1-2: 79-89.

[4] Cramer, E. and Kamps, U. 2003. Marginal distributions of sequential and generalized order statistics. Metrika. 53:3: 289-310.

[5] Cramer, E., Kamps, U. and Rychlik, T. 2002. Moments of generalized order statistics. In: H. Langseth and B. Lindqvist (Hrsg.). Third International Conference on Mathematical Methods in Reliability, Methodology and Practice. Trondheim: NTNU. S: 165-168.

[6] Cramer, E., Kamps, U. and Rychlik, T. 2002. On the existence of moments of generalized order statistics. Statist. and aprobability Letters. 59: 397-404.

[7] Cramer, E., Kamps, U. and Rychlik, T. 2002. Evaluations of expected generalized order statistics in various scale units. Applicationes Mathematicae. 29:3: 285-295. Garg, M. 2009. On generalized order statistics from Kumaraswami distribution. Tamsui Oxford Journal of Mathematical Sciences. 25:2: 153-166.

[8] Gupta, J. and Garg, M. 2018. The Lomax-Weibull distribution, Advanced Science Letters. 24 : 11: 8126-8129.

[9] Kamps, U. 1995. A concept of generalised order statistics. J. Statist. Plann. Inference. 48: 1-23.

[10] Kamps, U. and Cramer, E. 2001. On the distributions of generalized order statistics. Statist. 35: 269-280.